Canonical Bases for $U_q(\mathfrak{gl}(m|1))$

Cailan Li

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1 Canonical Bases: The Classical Case

Let $\mathfrak{g}$ be a simply laced of finite type with Weyl group W.

Theorem 1.1 (Lusztig). *Let $\vec{i}$ be a reduced expression for w_0. Let $\mathcal{L}_{\vec{i}} = \operatorname{span}_{\mathbb{Z}[q^{-1}]} B_{\vec{i}}$*

(i) The $\mathbb{Z}[q^{-1}]$ module $\mathcal{L}_{\vec{i}}$ is independent of $\vec{i}$.

(ii) Let $\pi : \mathcal{L}_{\vec{i}} \to \mathcal{L}_{\vec{i}}/q^{-1}\mathcal{L}_{\vec{i}}$. Then $\pi(B_{\vec{i}})$ is independent of $\vec{i}$.

Proof. We sketch the steps. Using the braid operators T_i one can reduce both statements to the rank 2 case where $\vec{i} = (i, j, i)$ $\vec{j} = (j, i, j)$, $i \cdot j = -1$.

(i) By definition,

$$\mathcal{L}_{\vec{j}} = \bigoplus_{(c_1,c_2,c_3)\in\mathbb{N}^3} \mathbb{Z}[q^{-1}]E_{\vec{j}}^{(c_1,c_2,c_3)} \qquad \text{where } E_{\vec{j}}^{(c_1,c_2,c_3)} = E_j^{(c_1)}(E_iE_j - q^{-1}E_jE_i)^{(c_2)}E_i^{(c_3)}$$

What Lusztig does is to show both $\mathcal{L}_{\vec{j}}$ equals $\mathcal{L} = \operatorname{span}_{\mathbb{Z}[q^{-1}]} CB$ where

$$CB = \left\{E_i^{(p)}E_j^{(q)}E_i^{(r)} \,|q \geq p + r\right\} \cup \left\{E_j^{(p)}E_i^{(q)}E_j^{(r)} \,|q \geq p + r\right\} \quad E_i^{(a)}E_j^{(b)}E_i^{(b-a)} = E_j^{(b-a)}E_i^{(b)}E_j^{(a)}$$

by showing that when $q \geq p + r$,

$$E_i^{(p)}E_j^{(q)}E_i^{(r)} = \sum_{n=0}^{p} a_n E_{\vec{j}}^{(q-n,n,p-n+r)} \qquad a_n \in \begin{cases} q^{-1}\mathbb{Z}[q^{-1}] & \text{if } n < p \\ 1 & \text{if } n = p \end{cases} \tag{1}$$

and similarly with i and j swapped which shows $\mathcal{L} \subseteq \mathcal{L}_{\vec{j}}$ and the other inclusion is clear. Now the trick is to note that the roles of i and j are symmetric so we will automatically have $\mathcal{L}_{\vec{i}} = \mathcal{L}$ as well.

(ii) We use the same strategy, showing that $\pi(B_{\vec{j}}) = \pi(CB)$ for any $\vec{j}$. Using Eq. (1) we see that when $q \geq p + r$

$$\pi(E_i^{(p)}E_j^{(q)}E_i^{(r)}) = \pi(E_{\vec{j}}^{(q-p,p,r)}), \qquad \pi(E_j^{(p)}E_i^{(q)}E_j^{(r)}) = \pi(E_{\vec{j}}^{(p,r,q-r)})$$

But notice that because $q - p \geq r$ we have that

$$\left\{\pi(E_{\vec{j}}^{(q-p,p,r)})\right\}_{q-p\geq r} = \left\{\pi(E_{\vec{j}}^{(a,b,c)})\right\}_{a\geq c}$$

and similarly one can check that

$$\left\{\pi(E_{\vec{j}}^{(p,r,q-r)})\right\}_{q-p\geq r} = \left\{\pi(E_{\vec{j}}^{(a,b,c)})\right\}_{a\leq c}$$

and thus $\pi(B_{\vec{j}}) = \pi(CB)$ for any $\vec{j}$ as desired. ∎

1.1 Bar Involution

Definition 1.2. *The bar involution* $\overline{}$ *on* $U_q(\mathfrak{g}^+)$ *is the* $\mathbb{Q}$ *algebra involution defined on generators by*

$$\overline{E_i} = E_i, \qquad \overline{q} = q^{-1}$$

Definition 1.3. *Let* $M = |\Phi^+|$. *Consider the total orders on* $\mathbb{N}^M$ $>_l$ *and* $>_r$ *where*

- $\mathbb{c} >_l \mathbb{d}$ *if* $c_1 > d_1$ *or* $c_1 = d_1$ *and* $(c_2, \ldots) >_l (c_2, \ldots)$, *etc.*
- $\mathbb{c} >_r \mathbb{d}$ *if* $c_M > d_M$ *or* $c_M = d_M$ *and* $(\ldots, c_{M-1}) >_r (\ldots, d_{M-1})$, *etc.*

Define the partial order $\mathbb{c} > \mathbb{d}$ *if* $\mathbb{c} >_l \mathbb{d}$ *and* $\mathbb{c} >_r \mathbb{d}$

Proposition 1.4. *For every reduced expression* $\vec{i}$ *we have that*

$$\overline{E_{\vec{i}}^{\mathbb{c}}} = E_{\vec{i}}^{\mathbb{c}} + \sum_{\mathbb{c}' > \mathbb{c}} r_{\mathbb{c}}^{\mathbb{c}'}(q) E_{\vec{i}}^{\mathbb{c}'}$$

where $r_{\mathbb{c}}^{\mathbb{c}'}(q)$ *are Laurent polynomials in* q.

Remark. The sum on the RHS above is finite, only $\mathbb{c}'$ in the same weight space as $\mathbb{c}$ can appear.

Theorem 1
For each reduced expression $\vec{i}$ *of* w_0 *there is a unique basis* $\left\{b_{\vec{i}}^{\mathbb{c}}\right\}_{\mathbb{c} \in \mathbb{N}^M}$ *of* $U_q(\mathfrak{g}^+)$ *contained in* $\mathcal{L}$ *such that*

(i) $\overline{b_{\vec{i}}^{\mathbb{c}}} = b_{\vec{i}}^{\mathbb{c}}$ *(self-duality)*

(ii) $b_{\vec{i}}^{\mathbb{c}} = E_{\vec{i}}^{\mathbb{c}} + \sum_{\mathbb{c}' > \mathbb{c}} a_{\mathbb{c}}^{\mathbb{c}'}(q) E_{\vec{i}}^{\mathbb{c}'}$ *where* $a_{\mathbb{c}}^{\mathbb{c}'}(q) \in q^{-1}\mathbb{Z}[q^{-1}]$ *for any* $\vec{i}$. *(degree bound)*

Moreover $CB_{\mathfrak{g}} := \left\{b_{\vec{i}}^{\mathbb{c}}\right\}_{\mathbb{c} \in \mathbb{N}^M}$ *is independent of* $\vec{i}$ *and is called the canonical basis of* $U_q(\mathfrak{g}^+)$.

Proof. Existence: Fix $\mathbb{c}$ minimal. Then Proposition 1.4 shows that $\overline{E_{\vec{i}}^{\mathbb{c}}} = E_{\vec{i}}^{\mathbb{c}}$ and thus we can set $b^{\mathbb{c}} = E_{\vec{i}}^{\mathbb{c}}$. Now for $\mathbb{c}$ non-minimal by Proposition 1.4 and induction one can write

$$\overline{E_{\vec{i}}^{\mathbb{c}}} = E_{\vec{i}}^{\mathbb{c}} + \sum_{\mathbb{c}' > \mathbb{c}} p_{\mathbb{c}}^{\mathbb{c}'}(q) b^{\mathbb{c}'}$$

where the $p_{\mathbb{c}}^{\mathbb{c}'}(q)$ are Laurent polynomials. Now by bar invariance of $b^{\mathbb{c}'}$ and Proposition 1.4 we see that

$$E_{\vec{i}}^{\mathbb{c}} = \overline{\overline{E_{\vec{i}}^{\mathbb{c}}}} = \left(E_{\vec{i}}^{\mathbb{c}} + \sum_{\mathbb{c}' > \mathbb{c}} p_{\mathbb{c}}^{\mathbb{c}'}(q) b^{\mathbb{c}'}\right) + \sum_{\mathbb{c}' > \mathbb{c}} p_{\mathbb{c}}^{\mathbb{c}'}(q^{-1}) b^{\mathbb{c}'} \implies p_{\mathbb{c}}^{\mathbb{c}'}(q) = -p_{\mathbb{c}}^{\mathbb{c}'}(q^{-1})$$

Because $p_{\mathbb{c}}^{\mathbb{c}'}(q)$ are Laurent polynomials it actually follows that

$$p_{\mathbb{c}}^{\mathbb{c}'}(q) = q^{-1} f_{\mathbb{c}}^{\mathbb{c}'}(q^{-1}) - q f_{\mathbb{c}}^{\mathbb{c}'}(q)$$

where $f_{\mathbb{c}}^{\mathbb{c}'}(q)$ is a polynomial. Now set

$$b_{\vec{i}}^{\mathbb{c}} = E_{\vec{i}}^{\mathbb{c}} + \sum_{\mathbb{c}' > \mathbb{c}} q^{-1} f_{\mathbb{c}}^{\mathbb{c}'}(q^{-1}) b_{\vec{i}}^{\mathbb{c}'}$$

By construction $b^{\mathbb{e}}$ satisfies (ii), and we compute

$$\overline{b^{\mathbb{e}}_{\vec{i}}} = \left(E^{\mathbb{e}}_{\vec{i}} + \sum_{\mathbb{e}'>\mathbb{e}} q^{-1} f^{\mathbb{e}'}_{\mathbb{e}}(q^{-1}) b^{\mathbb{e}'}_{\vec{i}} - \sum_{\mathbb{e}'>\mathbb{e}} q f^{\mathbb{e}'}_{\mathbb{e}}(q) b^{\mathbb{e}'}_{\vec{i}} \right) + \sum_{\mathbb{e}'>\mathbb{e}} q f^{\mathbb{e}'}_{\mathbb{e}}(q) b^{\mathbb{e}'} = b^{\mathbb{e}}_{\vec{i}}$$

and thus $b^{\mathbb{e}}_{\vec{i}}$ satisfies (i) as well.

Uniqueness: For each $\vec{i}$, $b^{\mathbb{e}}_{\vec{i}}$ is unique by the same argument as for KL basis, look at [EMTW] Chapter 3.

Independence of $\vec{i}$: For $\vec{j} \neq \vec{i}$ another reduced expression for w_0 notice

$$\pi(b^{\mathbb{e}}_{\vec{i}}) = \pi(E^{\mathbb{e}}_{\vec{i}}) \overset{\textit{Theorem 1.1}}{=\!=} \pi(E^{\mathbb{d}}_{\vec{j}}) = \pi(b^{\mathbb{d}}_{\vec{j}})$$

Because $\left\{b^{\mathbb{e}}_{\vec{i}}\right\}$ is unit triangular to $\left\{E^{\mathbb{e}}_{\vec{i}}\right\}$ it follows that $\left\{b^{\mathbb{e}}_{\vec{i}}\right\}$ is also a basis for $\mathcal{L}_{\vec{i}}$ and thus

$$b^{\mathbb{e}}_{\vec{i}} - b^{\mathbb{d}}_{\vec{j}} = \sum_{\mathbb{e}} h^{\mathbb{e}}(q) b^{\mathbb{e}}_{\vec{i}}, \qquad h^{\mathbb{e}}(q) \in q^{-1}\mathbb{Z}[q^{-1}]$$

The LHS is bar-invariant and so are the basis vectors on the RHS. This implies $h^{\mathbb{e}}(q) \in q^{-1}\mathbb{Z}[q^{-1}] \cap q\mathbb{Z}[q] = 0$ as desired. ■

Remark. The existence proof above also works for the KL basis, but the construction is more inefficient than the one in [EMTW].

Remark. Eq. (1) shows that $CB = CB_{\mathfrak{sl}_3}$. All of the above also works for $U_q(\mathfrak{g}^-)$ and we will also write $CB_{\mathfrak{g}}$ for the canonical basis of $U_q(\mathfrak{g}^-)$.

Corollary 1.5. *Let $\lambda \in \Lambda^+$ and let $\pi_\lambda : U_q(\mathfrak{g}^-) \to U_q(\mathfrak{g}^-)/I_\lambda = L(\lambda)$. Then*

$$B_\lambda = \{\pi_\lambda(b) \,|b \in CB_{\mathfrak{g}}, b \notin I_\lambda\}$$

is a basis for $L(\lambda)$.

Proof. Step 1: B_λ is a basis $\iff$ $CB_{\mathfrak{g}} \cap I_\lambda$ spans I_λ as a $\Bbbk$ v.s. We leave this as an exercise for the reader. Now write $\lambda = \sum c_i \omega_i$ as a sum of fundamental weights and note

$$I_\lambda = \sum_{j \in I} U_q(\mathfrak{g}^-) F_j^{c_j+1}$$

Thus we see that it suffices to show
Step 2: $U_q(\mathfrak{g}^-)F_j^{c_j+1} \in \operatorname{span}_{\Bbbk}\left\{CB_{\mathfrak{g}} \cap U_q(\mathfrak{g}^-)F_j^{c_j+1}\right\}$ $\forall j$. We first need a lemma

Lemma 1.6. *Let $\vec{i}$ be a reduced expression for w_0. Suppose that $\beta_t = s_{i_1} \cdots s_{i_{t-1}}(\alpha_{i_t}) = \alpha_k$ for $\alpha_k \in \Pi$ a simple root. Then we have that $F_{\vec{i},\beta_t} = F_k$.*

Now notice that

$$\beta_M = s_{i_1} \cdots s_{i_{M-1}}(\alpha_{i_M}) = s_{i_1} \cdots s_{i_{M-1}} s_{i_M}(-\alpha_{i_M}) = -w_0(\alpha_{i_M})$$

Note $-w_0 : \Phi^+ \to \Phi^+$ and because w_0 is linear it restricts to $-w_0 : \Pi \to \Pi$ and so $-w_0(\alpha_{i_M}) = \alpha_\ell$ for some ℓ. Thus using the lemma above it follows that

$$F_{\vec{i},\beta_M} = F_\ell$$

For each $j \in I$, set $\alpha_{k_j} = -w_0(\alpha_j)$. As w_0 is the longest element, we can always find a reduced expression $\overrightarrow{w}(j)$ for w_0 which ends in s_{k_j}, so that $\beta_M = -w_0(\alpha_{k_j}) = \alpha_j$. It then follows from above that

$$F_{\overrightarrow{w}(j),\beta_M} = F_j$$

And since $B_{\overrightarrow{w}(j)}$ is a basis for $U_q(\mathfrak{g}^-)$ it follows that

$$U_q(\mathfrak{g}^-)F_j^{c_j+1} \in \operatorname{span}_{\Bbbk}\left\{B_{\overrightarrow{w}(j)} \cap U_q(\mathfrak{g}^-)F_j^{c_j+1}\right\}$$

Let $\mathbb{e} = (\dots, e_M)$ $E^{\mathbb{e}}_{\overrightarrow{w}(j)} \in U_q(\mathfrak{g}^-)F_j^{c_j+1} = S_j$ (this means $e_M \geq c_j + 1$). We claim that $b^{\mathbb{e}}_{\overrightarrow{w}(j)} \in S_j$. Indeed since

$$b^{\mathbb{e}}_{\overrightarrow{w}(j)} = E^{\mathbb{e}}_{\overrightarrow{w}(j)} + \sum_{\mathbb{e}' > \mathbb{e}} a^{\mathbb{e}'}_{\mathbb{e}}(q) E^{\mathbb{e}'}_{\overrightarrow{w}(j)}$$

and by definition $\mathbb{e}' > \mathbb{e}$ means that $e'_M > e_M \geq c_j + 1$. Thus all elements on the RHS above are in S_j and so $b^{\mathbb{e}}_{\overrightarrow{w}(j)} \in S_j$. Because the change of basis matrix from $\left\{b^{\mathbb{e}}_{\overrightarrow{w}(j)}\right\}$ to $E^{\mathbb{e}}_{\overrightarrow{w}(j)}$ is upper triangular $\forall j$ and $\left\{b^{\mathbb{e}}_{\overrightarrow{w}(j)}\right\} = CB_{\mathfrak{g}}$ by Theorem 1 and thus

$$U_q(\mathfrak{g}^-)F_j^{c_j+1} \in \operatorname{span}_{\Bbbk}\left\{CB_{\mathfrak{g}} \cap U_q(\mathfrak{g}^-)F_j^{c_j+1}\right\} \qquad \forall j$$

■

Remark. In other words the fact that we had multiple PBW bases for $U_q(\mathfrak{g}^-)$ was a feature, not a bug of the theory.

2 The Super Case

2.1 PBW for $U_q(\mathfrak{gl}(m|1))$

In this section we only work with $U_q(\mathfrak{gl}(m|1))$.

Theorem 2 (Clark)
Let C be a super Cartan matrix for $U_q(\mathfrak{gl}(m|1))$ and set $D = s_i(C)$. Then define $T_i^s : U_q(C) \to U_q(D)$ as

$$T_i^s(E_{C,j}) = \begin{cases} -F_{D,i}K_{D,i} & \text{if } j = i \\ E_{D,i}E_{D,j} - (-1)^{p_D(i)p_D(j)} q^{D_{ij}} E_{D,j}E_{D,i} & \text{if } j \sim i \\ E_{D,j} & \text{if } j \not\sim i \end{cases}$$

We omit the definition for the other generators. Then T_i^s is a $\mathbb{Z}_2-$algebra isomorphism.

Proposition 2.1 (Clark). *The T_i^s satisfy braid relations of type A between appropriate $U_q(C)$, i.e. if $i \not\sim j$, given a super Cartan matrix B, let $C = s_i(B), D = s_j(C)$, then as maps $U_q(B) \to U_q(D)$ $T_i^sT_j^s = T_j^sT_i^s$, and similarly with $i \sim j$.*

Theorem 3 (Clark)
Fix Π for $\mathfrak{gl}(m|1)$ and let $C = C_\Pi$. Fix a reduced expression $\vec{i} = s_{i_1} \dots s_{i_K}$ for $w_0 \in S_{m+1}$. Define $\beta_t^\Pi = s_{i_1} \cdots s_{i_{t-1}}(\alpha_{i_t}^\Pi)$ and let $C_{\vec{i},t} = s_{t-1} \cdots s_{i_1}(C)$ (so $C_{\vec{i},1} = C$). Finally let

$$\begin{aligned} E_{\vec{i}:\beta_1^\Pi} &:= E_{C,i_1} \\ E_{\vec{i}:\beta_2^\Pi} &:= T_{i_1}^s(E_{C_{\vec{i},2},i_2}) \\ &\vdots \\ E_{\vec{i}:\beta_t^\Pi} &:= T_{i_1}^s \dots T_{i_{t-1}}^s(E_{C_{\vec{i},t},i_t}) \\ &\vdots \end{aligned}$$

and set

$$B_{\vec{i}}^\Pi = \left\{ E_{\vec{i}:\beta_1^\Pi}^{(a_1)} E_{\vec{i}:\beta_2^\Pi}^{(a_2)} \cdots E_{\vec{i}:\beta_L^\Pi}^{(a_L)} \,\middle|\, a_i \in \mathbb{Z}^{\geq 0}, a_s < 2 \text{ if } p(\beta_s^\Pi) = 1 \right\}$$

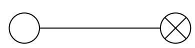
Then $B_{\vec{i}}^\Pi$ is a (PBW) basis for $U_q^+(C)$.

Remark. Because $E_{C_{\vec{i},t},i_t} \in U_q(C_{\vec{i},t}) = U_q(s_{t-1} \dots s_{i_1}(C))$ we see that

$$T_{i_1}^s \dots T_{i_{t-1}}^s(E_{C_{\vec{i},t},i_t}) \in U_q((s_{i_1} \dots s_{i_{t-1}})(s_{i_{t-1}} \dots s_{i_1})(C)) = U_q(C)$$

The miracle is that it's in fact in $U_q^+(C)$.

Example 1. For $U_q(\mathfrak{gl}(2|1))$ let $\Pi = \{\alpha_1, \alpha_2\}$ where α_2 is isotropic. $D(\Pi)$ will then be



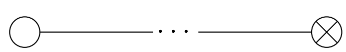
Let $E_{(12)} = E_1E_2 - q^{-1}E_2E_1$ and let $\vec{i} = s_1s_2s_1$, then

$$B_{\vec{i}}^\Pi = \left\{ E_1^{(r)} E_{(12)}^b E_2^a \,|\, 0 \leq a, b \leq 1,\ r \geq 0 \right\}$$

aka this is exactly the same as for $U_q(\mathfrak{sl}_3)$ except $a, b \leq 1$.

2.2 Canonical Bases: $U_q(\mathfrak{gl}(m|1))$ Standard

Theorem 2.2 (Clark)**.** *Let $\vec{i}$ be a reduced expression for w_0 and fix Π for $\mathfrak{gl}(m|1)$ to be the standard Borel, aka the decorated Dynkin diagram will be*



Let $\mathcal{L}_{\vec{i}}^\Pi = \text{span}_{\mathbb{Z}[q^{-1}]} B_{\vec{i}}^\Pi$

(i) The $\mathbb{Z}[q^{-1}]$ module $\mathcal{L}_{\vec{i}}^\Pi$ is independent of $\vec{i}$.

(ii) Let $\pi : \mathcal{L}_{\vec{i}}^\Pi \to \mathcal{L}_{\vec{i}}^\Pi / q^{-1}\mathcal{L}_{\vec{i}}^\Pi$. Then $\pi(B_{\vec{i}}^\Pi)$ is independent of $\vec{i}$.

Proof. Like in the classical case it suffices to do this for rank 2. For the standard Borel, there is only one isotropic root. As in Theorem 1.1 a key input for the proof is prior knowledge of what the canonical basis of $U_q^+(\mathfrak{gl}(2|1)_\Pi)$ is. [K]/[CHW3] writes this down as

$$CB_{\mathfrak{gl}(2|1)_\Pi} = \left\{ E_1^{(r)},\ E_1^{(r)}E_2,\ E_2E_1^{(r+1)},\ E_2E_1^{(r+1)}E_2 | r \geq 0 \right\}$$

[CHW3] then does the relevant computation to show these can be written as $\Bbbk$−linear sums of elements in $B_{\vec{i}}^\Pi$. ■

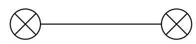
Corollary 2.3. *Let $\lambda \in \Lambda^+$ for $\mathfrak{gl}(m|1)$ and let Π be the standard Borel. Let $\pi_\lambda : U_q^-(\mathfrak{gl}(m|1)_\Pi) \to U_q^-(\mathfrak{gl}(m|1)_\Pi)/I_\lambda = K(\lambda)$ where $K(\lambda)$ is the Kac module of highest weight λ. Then*

$$B_\lambda = \left\{\pi_\lambda(b)\,|b \in CB_{\mathfrak{gl}(m|1)_\Pi}, b \notin I_\lambda\right\}$$

is a basis for $K(\lambda)$.

2.3 Canonical Bases: $U_q(\mathfrak{gl}(2|1))$ all isotropic

Here we have that $D(\Pi)$ is



Now when we construct the PBW bases $B_{\vec{i}}^{\Pi}$ and set $\mathcal{L}_{\vec{i}} = \text{span}_{\mathbb{Z}[q]} B_{\vec{i}}$, $\mathcal{L}_{\vec{i}}$ is dependent on $\vec{i}$!

Example 2. Let $\vec{i} = s_1 s_2 s_1$ and $\vec{j} = s_2 s_1 s_2$. We then compute

$$(B_{\vec{i}})_{2\alpha_1+2\alpha_2} = \left\{E_1E_2E_1E_2, \frac{E_2E_1E_2E_1}{[2]} + q^2\frac{E_1E_2E_1E_2}{[2]}\right\}$$
$$(B_{\vec{j}})_{2\alpha_1+2\alpha_2} = \left\{E_2E_1E_2E_1, \frac{E_1E_2E_1E_2}{[2]} + q^2\frac{E_2E_1E_2E_1}{[2]}\right\}$$

And thus

$$E_1E_2E_1E_2 = [2]\left(\frac{E_1E_2E_1E_2}{[2]} + q^2\frac{E_2E_1E_2E_1}{[2]}\right) - q^{-2}\left(E_2E_1E_2E_1\right)$$

and so we see that $E_1E_2E_1E_2$ is in neither the $\mathbb{Z}[q]$ or the $\mathbb{Z}[q^{-1}]$ span of $B_{\vec{j}}$.